NON-HAUSDORFF MANIFOLDS REVISITED

DAVID HOLMES

A topological manifold is sometimes defined as a second countable Hausdorff space that is locally homeomorphic to Euclidean space. We are interested in what happens when we drop the 'Hausdorff' condition. This motivates the following definition:

Definition 0.1. A *manifold* of dimension n is a second countable topological space which admits a cover by open sets U_i such that for each i there exists a open subset $S_i \subset \mathbb{R}^n$ and a homeomorphism $U_i \to S_i$.

A simple example is given by the 'real line with doubled origin' - this is obtained by taking two copies of the real line, and glueing them together along the complement of the origin. Manifolds arise naturally in several situations, such as the study of families of manifolds and vector bundles.

A topological space T is *compact* if every open cover of T admits a finite subcover. We are interested in the structure of compact manifolds. For example, we can make a compact non-Hausdorff manifold by making a 'circle with doubled point' in an analogous manner to the line with doubled origin.

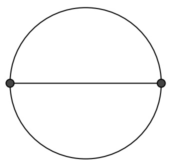
How 'nasty' can compact non-Hausdorff manifolds get? Last year, Anne Hommelburg worked on this topic (see http://www.math.leidenuniv.nl/en/theses/514/). Among other things, she showed by means of a counterexample a negative answer to the following question:

Question 0.2. *Let M be a compact manifold. Is it true that there exists a finite set of compact manifolds C_i and open immersions $C_i \to M$ such that the images of the C_i cover M?*

On the other hand, the examples she constructed do not feel morally too horrible. Thus we look for other ways in which compact manifolds might be considered 'nice'. In particular, we ask:

Question 0.3. *Let M be a compact manifold. Dos there exist a compact Hausdorff manifold M' and a continuous map $f\colon M' \to M$ such that f induces an isomorphism on every homotopy group?*

For example, if M is the 'circle with doubled origin', then let M' be a circle with a diameter line glued in (see below), and where the map squashes this extra line onto the circle except at the doubled point. How generally can you make constructions like this?



Date: December 11, 2014.